

On the ring of inertial endomorphisms of an abelian group

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Abstract. An endomorphisms φ of an abelian group A is said inertial if each subgroup H of A has finite index in $H + \varphi(H)$. We study the ring of inertial endomorphisms of an abelian group. We obtain a satisfactory description modulo the ideal of finitary endomorphisms. Also the corresponding problem for vector spaces is considered.

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1 Introduction and linear version

If φ is an endomorphism of an abelian group A and H a subgroup of A , then in [8] H is called a φ -inert subgroup iff H has finite index in $H + \varphi(H)$. Obviously, finite, finite index and φ -invariant subgroups are φ -inert. Passing to a “global condition”, we have -on one side- the notion of fully inert subgroup as in [9], that is φ -inert w.r.t. any endomorphism φ . Motivation for studying fully inert subgroups comes from the investigation of the dynamical properties of an endomorphism of an abelian group (see [6], [8]). Fully inert subgroups of an abelian group A have been recently studied in [7], [9], [11], [12] in cases when A is a divisible group, a free group, a torsion-free module over the ring of p -adic integers, a p -group, resp.

On the other side, we call *inertial an endomorphism with respect to which each subgroup is inert*. This property has been studied in [3] and [4] in connection with the study of inert subgroups of groups (see [1], [10], [14]). Recall that in non-abelian group context, a subgroup is said to be *inert* if it is commensurable with its conjugates (that is w.r.t. inner automorphisms), where

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two subgroups H and H_1 of a group are told commensurable iff $H \cap H_1$ has finite index in both H and H_1 .

Clearly the definition of inertial endomorphism of an abelian group can be regarded as a local condition generalizing both notions of power and finitary endomorphism. Recall that an endomorphism is *power* if it maps each subgroup to itself and *finitary* if it has finite image.

In [4] we have proved the following Fact (and given a fruitful description of inertial endomorphisms as reported in the Characterization Theorem below).

Fact *Inertial endomorphisms of any abelian group A form a subring $IE(A)$ of the full ring $E(A)$ of endomorphisms of A . It contains the ideal $F(A)$ of finitary endomorphisms and many "multiplications" (see below). Moreover inertial endomorphisms of an abelian group commute modulo finitary ones, that is $IE(A)/F(A)$ is a commutative ring.*

In this paper we study the ring $IE(A)$ of inertial endomorphisms, while the group of invertible inertial endomorphisms has been studied in [5] along the same lines as in [15] and [2].

We first note that the corresponding problem for vector spaces has an easy solution.

Theorem 1 *Let φ be an endomorphism of an infinite dimension vector space V . Then each subspace H has finite codimension in $H + \varphi(H)$ iff φ acts as a scalar multiplication on a finite codimension subspace of V .*

On the other hand, for each subspace H the codimension of $H \cap \varphi(H)$ in H is finite iff φ acts as multiplication by a non-zero scalar on a finite codimension subspace.

Corollary 1 *The ring of endomorphisms φ of a vector space V such that each subspace H has finite codimension in $H + \varphi(H)$ is the sum of the ring of scalar multiplications and the ring of finitary endomorphisms.*

These statements apply also to the above defined ring $IE(A)$ in the case A is elementary abelian, clearly. A corresponding statement holds for "elementary" inertial endomorphisms (see section 3). To treat the periodic case, we introduce *mini-multiplications*, see sections 1 and 5. In the general case, the picture can be rather complicated, see our Main Result Theorem 2. Further relevant instances of (uniform) inertial endomorphisms are introduced in section 6 by Proposition 3 and Corollary 2.

Proof of Theorem 1 We use an argument similar to one we used in the proof of Proposition 5 of [4]. Fix φ and suppose that either for each $H \leq A$ the group $H + \varphi(H)/H$ is finite or for each $H \leq A$ the group $H/H \cap \varphi(H)$ is finite. By contradiction, assume φ is scalar multiplication on no quotient by a finite dim subspace. We first prove that *if W is a finite dim subspace, then W is contained in a finite dim φ -invariant subspace*. Clearly we can assume that $W = Ka$ has dimension 1. Consider then the homomorphism

$$\Phi : K[x] \mapsto V$$

mapping $1 \mapsto a$ and $x \mapsto \varphi(a)$. If Φ is injective, we can replace $\text{im}\Phi$ by $K[x]$ and φ by multiplication by x . Then both $H := K[x^2]$ and $\varphi(H) = xH$ have infinite dimension, while $H \cap xH = 0$, a contradiction. Therefore Φ is not injective and $\text{im}\Phi$ is the wished subspace.

Now we can prove that: *for all finite dimension subspaces $W \leq V$ such that $W \cap \varphi(W) = 0$ there exists a subspace $W' > W$ with finite dimension such that*

$$W' \cap \varphi(W') = 0 \quad \text{and} \quad \varphi(W') > \varphi(W).$$

To see this note that if $Z \geq W$ is a finite dim φ -invariant subspace, as φ does not act as a scalar multiplication on A/Z , we can choose $a \in V$ such that $\varphi(a) \notin Ka + Z$ and define $W' := Ka + W$. If now $b \in W' \cap \varphi(W')$, then $\exists h, k \in K, \exists c, c_0 \in W$ such that $b = ha + c = k\varphi(a) + \varphi(c_0)$. Thus $k\varphi(a) \in Ka + Z$ while $\varphi(a) \notin Ka + Z$. Therefore $k = h = 0$. It follows $b = c = \varphi(c_0) \in W \cap \varphi(W) = 0$, as claimed.

Finally, starting at $W_0 = 0$, by recursion we define $W_{i+1} := W'_i$ and $W_\omega := \bigcup_i W_i$. We get that both W_ω and $\varphi(W_\omega)$ have infinite dimension and $W_\omega \cap \varphi(W_\omega) = 0$, the wished contradiction. The statement follows now easily. \square

2 Notation and statement of Main Result

To state our main result, we need to consider some relevant invariants of the group and to introduce some *ad hoc definitions*. Note that in this paper A always stands for an abelian group (in additive notation). For undefined notation on abelian groups we refer to [13]. In particular we denote by \mathbb{Q}_p^* the ring of p -adic integers and by \mathcal{J} the ring $\prod_p \mathbb{Q}_p^*$ which is meant to act componentwise on any periodic abelian group. For $m, n \in \mathbb{Z}$, if we consider the fraction $m/n \in \mathbb{Q}$, we always mean m and $n \neq 0$ are coprime. We write $\pi(n)$ for the set of prime divisors of $n \in \mathbb{N}$ and π' for the complement of a set π of primes. Denote by A_π the π -component of A and $A[n] := \{a \in A \mid na = 0\}$. As usual, if $n \in \mathbb{N}$ annihilates A , that is $nA = 0$, we say that A is *bounded* by n . Further, we say that A is bounded, if it is bounded by some n and the least such n is called *the bound* of A . If $pA = A$ (resp. A is periodic and $A[p] = 0$) for each $p \in \pi$ say that A is π -divisible (resp. a π' -group).

We recall that in [4], we called *multiplications* of an abelian group A either the actions on A of a subring of \mathbb{Q} or, when A is periodic, the above action of \mathcal{J} . *Multiplications form a ring* $M(A)$. If A is periodic, clearly $M(A) \simeq \prod_p \mathbb{Q}_p^*/(p^{e_p})$, where the product is taken on all primes and p^{e_p} is the bound of the p -component A_p of A or 0 if A_p is unbounded. If A is non-periodic, then $M(A) \simeq \mathbb{Q}^\pi$ where π is the largest set of primes such that A is a \mathbb{Q}^π -module, that is A is p -divisible with no elements of order p , $\forall p \in \pi$. Here, as usual, \mathbb{Q}^π is the ring of rationals whose denominator is a π -number, that is divided by primes in π only.

Recall also that if A is any abelian group there is a free abelian subgroup F such that A/F is periodic. The rank of F coincides with the torsion-free rank

$r_0(A)$, that is the rank of the torsion free group A/T , where $T = T(A)$ denotes the torsion subgroup of A , as usual. In Proposition 1 of [4] we noticed that *when A is not periodic multiplications which are not by an integer are inertial iff the underlying abelian group A has finite torsion-free rank*. For abelian groups with infinite torsion-free rank, case (a) in Characterization Theorem below. Thus we will be mainly concerned with groups with finite torsion-free rank, say FTFR.

When A has FTFR all subgroups F as above are commensurable. Fix one of them and define the *sequence of essential p -bounds* (ϵ_p) such that ϵ_p is the min i such that $\sum_{j>i} f_j < \infty$ where f_j is the *Ulm-Kaplanski invariant* of the p -component $(A/F)_p$ and $\epsilon_p = \infty$ if the p -component of A/F is unbounded. Clearly p runs on all primes. In other words, $\epsilon_p = \infty$ or ϵ_p is the smallest e such that $p^e(A/F)_p$ is finite. Clearly the sequence (ϵ_p) is independent of F .

We also consider the *sequence of p -bounds* (e_p) , where p^{e_p} is the bound of $(A/F)_p$, when this is bounded, or $p^{e_p} = \infty$, otherwise. This sequence depends on the choice of F . Clearly $\epsilon_p \leq e_p$ for each prime p . However sequences (e_p) 's corresponding to different F are definitely equal and coincide on entries which are infinite. Then we may consider an equivalence relation such that the class of the sequence (e_p) depends on A only.

Thus we define (independently of F and of the appearing e_p 's) the ring

$$\mathcal{H}(A) := \frac{\prod_p Q_p^*/(p^{e_p})}{\bigoplus_p (p^{e_p})/(p^{e_p})} \simeq \frac{\prod_p Q_p^*}{\bigoplus_p (p^{e_p}) + \prod_p (p^{e_p})}$$

where (e_p) and (ϵ_p) are sequences of p -bounds and essential p -bounds resp, (p^x) is ideal generated by p^x in the ring Q_p^* , where $p^\infty := 0$. Also denote by:

- $\pi_*(A)$ the set of primes p such that A_p is bounded and A/A_p is p -divisible.
- $\pi_c(A)$ the set of primes for which for some F as above the p -component of A/F is *critical*, that is with shape $B \oplus D$ with B infinite but bounded and $D \neq 0$ divisible with finite positive rank. Clearly $\pi_c(A)$ is independent of F , as A has FTFR.

As we shall often face some particular decomposition of an abelian group let us introduce further notation. If $A = B \oplus C$ and there is a *finite* set π of primes, such that B is a *bounded* π -subgroup and the π -component C_π of C is *divisible with finite rank* we write

$$A = B \oplus_\pi C$$

and call *mini-multiplications* endomorphisms of shape $n \oplus 0$ (i.e. acting as $n \in \mathbb{Z}$ on B and annihilating C). For details see section 1)

Further, it is easy to verify that *if $\pi \subseteq \pi_*(A)$, then $A = B \oplus_\pi C$ for $B = A_\pi$ and subgroup C is a \mathbb{Q}^π -module which is uniquely determined and fully invariant*, clearly. In such a condition, we say that $\varphi := n \oplus_\pi \alpha$ is a *semi-multiplication* where φ is the multiplication by $n \in \mathbb{Z}$ on B and by α on C , where either $\alpha \in \mathcal{J}$ or $\alpha \in \mathbb{Q}^\pi$ according to C is periodic or not. Denote by $SM(A)$ the ring formed by the semi-multiplications. It is *a central subring of $E(A)$* .

Finally, say that an inertial endomorphism φ is *uniform* iff there is a φ -series $F \leq A_0 \leq A$ such that A/A_0 is finite, φ is multiplication on the periodic group A_0/F and $\varphi = 0$ on the free abelian subgroup F . Denote by $UI(A)$ the *subring of uniform inertial endomorphisms* (see section 6).

Now we can describe the ring $IE(A)$ of inertial endomorphisms in terms of above invariants and others that we will introduce below. Symbols of (direct) sum are to be read in the additive group of $IE(A)$.

Theorem 2 (Main Theorem) *Let A be an abelian group.*

- a) *If A has not FTFR, then $IE(A) = M(A) \oplus F(A)$, where $M(A) \simeq \mathbb{Z}$ is the ring of multiplications by integers.*
- b) *If A has FTFR, then*

$$IE(A) = SM(A) + UI(A) + N$$

where:

- $SM(A)$ is the (central) subring of semi-multiplications of A ;
- $UI(A)$ is the subring of uniform inertial endomorphisms of A ;
- $N \simeq \bigoplus_{p \in \pi_c(A)} \mathbb{Z}(p^{c_p})$ is a subring of inertial mini-multiplications of A .

Further, we have:

- 1) $UI(A) + N$ is the ideal of inertial endomorphisms of A with periodic image;

$$2) \frac{IE(A)}{UI(A) + N} \simeq \mathbb{Q}^{\pi_*(A)} \text{ as rings (provided } A \text{ is non-periodic);}$$

$$3) \frac{UI(A)}{F(A)} \text{ is isomorphic to a subring of } \mathcal{H}(A) \text{ .}$$

c) In particular, if A is periodic, then:

- i) $IE(A) = M(A) + F(A) + N$;
- ii) $SM(A) = M(A)$ and $UI(A) = M(A) + F(A)$;
- iii) $\frac{UI(A)}{F(A)} \simeq \mathcal{H}(A)$ as rings.

For details on above N and c_p 's see Lemma 1. By Corollary 2 we will see that $UI(A)$ may be rather large even if A is countable. Now we recall a characterization of inertial endomorphisms (see Theorem A and Proposition 5 in [4]).

Characterization Theorem ([4]). *Let $\varphi_1, \dots, \varphi_l$ finitely many endomorphisms of an abelian group A . Then each φ_i is inertial if and only if there is a finite index subgroup A_0 of A such that one of (a) or (b) holds: (a) each φ_i acts as multiplication on A_0 by $m_i \in \mathbb{Z}$;*

(b) $A_0 = B \oplus D \oplus C$ and exist finite sets of primes $\pi \subseteq \pi_1$ such that:

i) $B \oplus D$ is the π_1 -component of A_0 where B is bounded, D divisible π' -group with finite rank.

ii) C is a $\mathbb{Q}^\pi[\varphi_1, \dots, \varphi_l]$ -module, with a submodule $V \simeq \mathbb{Q}^\pi \oplus \dots \oplus \mathbb{Q}^\pi$ (finitely many times) such that C/V is a π_1 -divisible π' -group.

iii) each φ_i acts by (possibly different) multiplications on $B, D, V, C/V$ where φ_i is represented by $m_i/n_i \in \mathbb{Q}$ on V and on all p -components of D such that the p -component of C/V is infinite and $\pi = \pi(n_1 \dots n_l)$.

Moreover, if A is periodic, then $\varphi_1, \dots, \varphi_l$ are inertial iff (FS) there is $m \in \mathbb{N}$ such that for each $X \leq A$ there are subgroups X_*, X^* which are φ_i -invariant ($\forall i$) and such that $X_* \leq X \leq X^* \leq A$ and $|X_*/X^*| \leq m$.

Clearly if A is torsion-free inertial then endomorphisms are multiplication by rationals. When A is periodic we have $V = 0$ and in particular:

- if A divisible and periodic then inertial endomorphisms are multiplication,
- if A is reduced and periodic inertial endomorphisms are multiplication on a finite index subgroup of A .

3 The ring $FM(A)$ of multiplication-by-finite endomorphisms

In this section, by Theorem 3 we study $FM(A)$, a relevant subring of $IE(A)$, which might have interest in itself, as well. The following is easy to check.

Fact If $\varphi \in E(A)$ acts as an inertial endomorphism on a finite index subgroup of A then φ is inertial on the whole A indeed. The same happens arguing modulo a finite φ -invariant subgroup.

We say that two endomorphisms are *close* iff their difference is finitary. We generalize Proposition 4 of [4] to non-periodic groups and give a picture of the ring of endomorphisms which are close to multiplications. We give a definition.

Denote by $\pi_0(A)$ the set of primes p such that A_p is finite and A/A_p is p -divisible. Clearly $\pi_0(A) \subseteq \pi_*(A)$. If π is a finite subset of $\pi_0(A)$ then $A = A_\pi \oplus_\pi C$. Notice that summands are fully invariant and uniquely determined. Call *quasi-multiplication* of a non-periodic group A those endomorphisms with shape $r \oplus_\pi m/n$ (with $r \in \mathbb{Z}$ and $m/n \in \mathbb{Q}^\pi$). Clearly these form a subring $QM(A)$ of $E(A)$. If A is periodic set $QM(A) := M(A)$. In any case, $QM(A) \subseteq SM(A)$ and $QM(A)$ is also in the center of $E(A)$.

Theorem 3 For an endomorphism φ of an abelian group A the following are equivalent:

- MF) φ acts by means of a multiplication on a finite index subgroup A_0 of A ,
- FM) φ acts by means of a multiplication modulo a finite subgroup A_1 of A .

Moreover, endomorphisms with the above properties form a subring

$$FM(A) = F(A) + QM(A)$$

which is contained in $IE(A)$, provided A has FTFR. Moreover,

- i) if A is non-periodic, then $FM(A)/F(A)$ is naturally isomorphic to the ring $\mathbb{Q}^{\pi_0(A)}$ and $F(A) \cap QM(A)$ consists of maps of type $r \oplus_\pi 0$ for a finite subset π of $\pi_0(A)$,
- ii) if A is a p -group, then $QM(A) = M(A)$; if A is unbounded it holds $F(A) \cap M(A) = 0$; otherwise, if $e < \infty$ and ϵ are the p -bound and the essential p -bound of A , resp., there is a natural ring isomorphism $F(A) \cap M(A) \simeq p^\epsilon \mathbb{Z}/p^e \mathbb{Z}$.

Note that an endomorphisms of a periodic abelian group A is (FM) iff it acts this way on finitely many components and by multiplications on all remaining

ones, clearly. Note also that on $A = \mathbb{Z}(p) \oplus_p \mathbb{Q}^{\{p\}}$ we have $0 \oplus_p 1/p \in FM(A) \setminus (F(A) + M(A))$.

Proof. $(MF) \Rightarrow (FM)$ If A is periodic, the statement is clear as there is $\alpha \in \mathcal{J}$ such that $\varphi = \alpha$ on A_0 and we can consider $A_1 := im(\varphi - \alpha)$. Otherwise, $\varphi = m/n \in \mathbb{Q}$ on A_0 , a \mathbb{Q}^π -module with $\pi := \pi(n)$. Thus A_π is finite and we may assume $A_\pi = 0$. Then note that A/T is torsion free with a finite index π -divisible subgroup $A_0 + T/T$. Then A/T is π -divisible. Therefore A is \mathbb{Q}^π -module. Again, $A_1 := im(\varphi - m/n)$ is the wished subgroup.

$(FM) \Rightarrow (MF)$ If A is non-periodic and $\varphi = m/n$ on the \mathbb{Q}^π -module A/A_1 (again $\pi := \pi(n)$), then $A_\pi \leq A_1$ is finite and there a \mathbb{Q}^π -module $A'_0 \leq A$ such that $A = A_\pi \oplus A'_0$ and one can consider the endomorphism $\varphi_0 := 0 \oplus m/n \in E(A)$ such that $im(\varphi - \varphi_0) \leq A_1$. Thus $A_0 := A'_0 \cap ker(\varphi - \varphi_0)$ has finite index in A . On the other hand since A'_0 is π -divisible and A'_0/A_0 is finite we have A'_0/A_0 is coprime to n and so A_0 is n -divisible, thus φ_0 -invariant. Therefore A_0 is φ -invariant too and is the wished subgroup. The periodic case is clear, with $\varphi = \alpha \in \mathcal{J}$ on A/A_1 and $\varphi_0 = \alpha \in M(A)$.

Moreover, let φ_i act by multiplication on A/A_i ($i = 1, 2$) with $A_3 := A_1 + A_2$ finite. As any subgroup of $T(A)/A_i$ is φ_i -invariant ($i = 1, 2$), such is A_3 . Then $\varphi_1 - \varphi_2$ and $\varphi_1\varphi_2$ act as multiplications on A/A_3 .

Above arguments also show that $FM(A) = F(A) + QM(A)$ as $\varphi_0 \in QM(A)$. From the Characterization Theorem above it follows that $FM(A) \subseteq IE(A)$ when A has FTFR.

Finally, when A is non-periodic, the map $\varphi = r \oplus m/n \in FM(A) \mapsto m/n \in \mathbb{Q}^{\pi_0(A)}$ is the wished isomorphism and if $r \oplus_\pi m/n \in QM(A)$ is finitary, then it is 0 on $A/T(A)$. Hence $m = 0$. On the other hand, when A is a p -group, if $0 \neq \alpha \in M(A) \cap F(A)$ we have that there exists i such that $ker \alpha = A[p^i]$ (clearly p^i is the maximal power of p dividing α). If $A[p^i]$ has finite index in A , then A is bounded and $\epsilon \leq i$. Conversely, if p^ϵ divides α it is plain that $\alpha \in F(A)$. \square

4 Mini-multiplications of an abelian group

From the introduction section recall the following:

Definition A *mini-multiplication* of an abelian group $A = B \oplus_\pi C$ is an endomorphism of shape $n \oplus 0$ i.e. acting as multiplication by $n \in \mathbb{Z}$ on the π -bounded subgroup B and annihilating C , where the π -component of C is divisible with finite total rank.

By next statement we consider the isomorphism type $NM(A)$ of the subring N which appears in the statement of Theorem 2. It is the type of a ring of multiplications with bounded support of a fully-invariant distinguished periodic section of A as well.

Lemma 1 Let A be an abelian group with FTFR and $p_i^{c_i}$ be the bound of $A_{p_i}/div(A_{p_i})$ for each i where $\pi_c(A) =: \{p_1, \dots, p_i, \dots\}$ is the set of critical primes of A . Let $\pi_i := \{p_1, \dots, p_i\}$.

There are sequences (B_i) and (C_i) of subgroups such that B_i is a bounded p_i -group $\forall i$ and

$$(*) \quad A = (B_1 \oplus \dots \oplus B_i) \oplus_{\pi_i} C_i \quad \text{and} \quad C_i = B_{i+1} \oplus C_{i+1}.$$

Fixed above sequences, mini-multiplications $n \oplus_{\pi_i} 0$ acting as $n \in \mathbb{Z}$ on $B_1 \oplus \dots \oplus B_i$ and 0 on C_i (for some i) form a ring N of inertial endomorphisms isomorphic to

$$NM(A) := \bigoplus_{p_i \in \pi_c(A)} \mathbb{Z}(p_i^{c_i}).$$

Proof. Note that the p_i -component of A is the sum of a bounded subgroup and a divisible one. Therefore A splits on it. We define inductively wished sequences (B_i) and (C_i) by (choosing) $A = B_1 \oplus C_1$ and $C_i = B_{i+1} \oplus C_{i+1}$. Then define mini-multiplications as in the statement. They are inertial by the Characterization Theorem. Notice that unfortunately they depend on the choice of the two sequences. Then for each i and coset class $n_i \in \mathbb{Z}(p_i^{c_i})$ consider the mini-multiplication which is multiplication by n_i on B_i and 0 on other summands. This gives the wished isomorphism. \square

Let us highlight the role of mini-multiplications. Say that an endomorphism ψ of an abelian group A is *bounded* iff $\psi(A)$ is bounded.

Lemma 2 *Let μ be a bounded multiplication of an abelian A . If A is non-periodic, then $\mu = 0$. If A is a p -group A , then either $\mu = 0$ or A is bounded.*

Proof. If A is non-periodic, consider the action of μ on $A/T(A)$. Otherwise assume $\mu \neq 0$ and write $\mu = p^r \alpha$ where α is an invertible p -adic and $r \in \mathbb{N}_0$ and consider that $p^r A = \mu(A)$, whence A is bounded. \square

Lemma 3 *Let φ be an endomorphism of A . If φ acts as a mini-multiplication of type $m \oplus_{\pi} 0$ on a subgroup of finite index A_0 of A , then φ is close to a mini-multiplication of type $m \oplus_{\pi} 0$ on the whole of A .*

Proof. Let $\varphi = m \oplus_{\pi} 0$ on $A_0 = B_0 \oplus_{\pi} C_0$. Then $A_{\pi} = B \oplus D$, where $D \leq C_0$ is divisible of finite rank and $B \geq B_0$ is bounded. Hence $A = A_{\pi} \oplus C_1$ and $C := D \oplus C_1$ is commensurable to C_0 . It is now clear that φ is close to the mini-multiplication $m \oplus_{\pi} 0$ on $A = B \oplus_{\pi} C$. \square

Proposition 1 *Let A be an abelian group.*

- a) *If A has not FTFR, bounded inertial endomorphisms are finitary.*
- b) *If A has FTFR, any bounded inertial endomorphism is the sum of a mini-multiplication and a finitary endomorphism.*

Proof. Let φ be a bounded inertial endomorphisms of A . If A has not FTFR, then φ is multiplication on a finite index subgroup of A and the statement follows as in Lemma 2.

If A has FTFR then we are in case (b) of the Characterization Theorem above. We use the same notation. Then $\varphi = m \in \mathbb{Z}$ on B and $\varphi = 0$ on D , as this is divisible. Let π_2 be the set of primes p in $\pi(\varphi(C))$ (which is finite) such that the p -component of C is bounded. Then $C = C_{\pi_2} \oplus C_1$. Also $\varphi = 0$ on

V . Moreover $C_1 \cap V$ is φ -invariant and $C/(C_1 \cap V)$ is periodic. Then by (FS), there is a finite index φ -invariant subgroup C_0 of C_1 .

Note that if $p \in \pi(\varphi(C_0))$ then $p \notin \pi_2$, hence $\varphi = 0$ on the (unbounded) p -component of C_0 , by Lemma 2. Therefore $\varphi = 0$ on the whole $T(C_0)$. Then $\varphi(C_0)$ is an image of $C_0/T(C_0)$ which has finite rank. As $\varphi(C_0)$ is bounded, it is even finite. Thus $\varphi = 0$ on a finite index subgroup C' of C_0 . Note that C' is a π_2 -group and C'_{π_1} is divisible with finite rank as C_{π_1} is. Thus φ is mini-multiplication on $(B + C_{\pi_2}) \oplus_{\pi_1 \cup \pi_2} C'$, which has finite index in A . By Lemma 3, φ is close to a mini-multiplication. \square

5 Periodic case

The ring $IE(A)$ when A is a p -group is described by the following result, which follows from the Characterization Theorem above. For the general periodic case note that an *endomorphism of an abelian torsion group is inertial iff it is such on all primary components and multiplication on all but finitely many of them*.

Recall that we say that an abelian p -group is *critical* when $\pi_c(A) = \{p\}$, that is when the maximum divisible subgroup $D := D(A)$ has positive finite rank and A/D is bounded but infinite.

Proposition 2 *Let A be an abelian p -group.*

- i) *If A is non-critical then $IE(A) = FM(A)$.*
- ii) *If A is critical and $D := D(A)$ then*

$$IE(A) = FM(A) + N = M(A) + F(A) + N$$

where $N \simeq M(A/D) \simeq \mathbb{Z}(p^e)$ is a subring of mini-multiplications and p^e is the bound of A/D .

Moreover $M(A) \cap (F(A) + N) = 0$ and $FM(A) \cap N = F(A) \cap N = p^\epsilon N$ where ϵ is the essential p -bound of A/D .

Proof. Let $\varphi \in IE(A)$. If A is non-critical, $\varphi \in FM(A)$ by the Characterization Theorem. If A is critical, fix a decomposition $A = B \oplus_p D$ with B bounded and $D = D(A)$ with finite rank. By the same theorem there is a φ -invariant finite index subgroup B_0 of B such that φ acts as multiplication by some $n \in \mathbb{Z}$ on a finite index subgroup C of B_0 .

For each p -adic $\beta \in Q_p^*$ denote by $\bar{\beta}$ the mini-multiplication $\bar{\beta} = \beta \oplus 0$ on $A = B \oplus_p D$. Then if $\alpha \in Q_p^*$ represents the action of φ on D (which is multiplication) we have $C + D \subseteq \ker(\varphi - \alpha - \overline{(n - \alpha)})$ and so $IE(A) = M(A) + F(A) + N$, where $N = \{\bar{n} \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}(p^e)$.

Further, if $\alpha = \varphi_0 + \bar{n} \in M(A) \cap (F(A) + N)$ (where $\varphi_0 \in F(A)$ and $\bar{n} \in N$), then $\alpha = 0$ on D . It follows $\alpha = 0$ on the whole of A . By a similar argument $FM(A) \cap N = F(A) \cap N$. Finally, if the mini-multiplication $m \oplus 0$ on the decomposition $A = B \oplus D$ is finitary, then $m = 0$ on $B[p^\epsilon]$ and p^ϵ divides m . \square

Recall that the description of group of units of $IE(A)$ given in [5] is rather complete when A is periodic.

6 The ring $UI(A)$ of uniform inertial endomorphisms

We consider now a further relevant ring of inertial homomorphisms, which contains $F(A)$.

Definition An inertial endomorphism φ is *uniform and represented by* $\alpha \in \mathcal{J}$ on A_0/F iff

- A/A_0 is finite
- A_0/F is periodic and φ is multiplication by α , on A_0/F .
- F is free abelian and $\varphi = 0$ on F .

Clearly, we are considering precisely those *inertial endomorphisms* φ which are FM on some A/F and have periodic image (which is not necessarily finite as in Proposition 2). Observe that such an α is not uniquely determined.

Proposition 3 *Let A be an abelian group with finite torsion-free rank. Then the set $UI(A)$ of uniform inertial endomorphisms of A is a subring of $E(A)$ containing $F(A)$, where:*

- i) $UI(A)/F(A)$ is isomorphic to a subring of $\mathcal{H}(A)$,
- ii) if A is periodic, $UI(A) = FM(A) = M(A) + F(A)$ and $UI(A)/F(A) \simeq \mathcal{H}(A)$,
- iii) there exists a periodic quotient \bar{A} of A such that $UI(\bar{A})/F(\bar{A}) \simeq \mathcal{H}(A)$.

Proof. For the definition of $\mathcal{H}(A)$ see the introduction. Clearly $F(A) \subseteq UI(A) \subseteq IE(A)$. Consider then the relation \mathcal{R} between $\varphi \in UI(A)$ and $\alpha \in \mathcal{J}$ defined by the following: φ is uniform and is represented by α on some A_0/F .

We claim that \mathcal{R} is compatible with ring operations. In fact if φ_i is represented by $\alpha_i \in \mathcal{J}$ on A_i/F_i ($i = 1, 2$), then $\varphi := \varphi_1 - \varphi_2$ acts as multiplication by $\alpha := \alpha_1 - \alpha_2$ on A_3/F_3 with $A_3 = (A_1 \cap A_2)$ and $F_3 := (F_1 + F_2) \cap A_3$. Hence φ is uniform and represented by α on A_0/F , where $F := F_1 \cap F_2$ and $A_0/F := \ker(\varphi - \alpha)|_{A/F}$ (note that F is free, A/F is periodic and $\varphi(F) = 0$, as all A_i 's and F_i 's are commensurable, resp.). For the multiplicative ring operation argue the same way. In particular we have that $UI(A) = \text{dom } \mathcal{R}$ is a subring. Analogously, $\text{cod } \mathcal{R}$ is a subring of \mathcal{J} . In the case A is periodic, it is plain that $\text{cod } \mathcal{R} = \mathcal{J}$.

On one hand we have $\{\varphi \in UI(A) \mid \varphi \mathcal{R} 0\} = F(A)$; in fact it is plain that if $\varphi \in F(A)$, then $\varphi \mathcal{R} 0$. Similarly, if $\varphi \mathcal{R} 0$, then $\varphi = 0$ on some A_0/F and on A/T as well. Therefore $\varphi = 0$ on A_0 (as $F \cap T = 0$) and $\varphi \in F(A)$.

Fix now F , related sequences (e_p) , (ϵ_p) and note that the following ideal does not depend on the choice of F

$$I := \bigoplus_p (p^{\epsilon_p}) + \prod_p (p^{e_p})$$

Let us show that $I = \{\alpha \in \mathcal{J} \mid 0 \mathcal{R} \alpha\}$. If $0 \mathcal{R} \alpha$ then p^{ϵ_p} divides α_p for all p and even p^{e_p} divides α_p for all but finitely many p . Thus $\alpha \in I$. Conversely,

let $\alpha = (\alpha_p) \in \prod_p(p^{e_p})$. Then $0\mathcal{R}\alpha$ since $\alpha = 0$ on A/F (as the p -component of A/F is bounded by p^{e_p}). Similarly if $\alpha \in \bigoplus_p(p^{e_p})$, let A_0/F be generated by the p^{e_p} socle of A/F for all p such that $\alpha_p \neq 0$ and the whole p -component of A/F for the remaining p . Thus $0\mathcal{R}\alpha$ as $\alpha = 0$ on A_0/F and A_0 has finite index in A .

From what we proved above, it follows clearly that \mathcal{R} induces an isomorphism as in the statement (i). For statement (ii) see Proposition 2.

For statement (iii), for each p choose an $L_p \geq F$ such that factor A/L_p is either of type $\mathbb{Z}(p^\infty)$ if $p^{e_p} = \infty$ or an infinite homogeneous p -group with bound p^{e_p} plus a group of type $\mathbb{Z}(p^{e_p})$, otherwise. Then set $\bar{A} := A/\bigcap_p L_p$ and get $\mathcal{H}(\bar{A}) = \mathcal{H}(A)$. Then (iii) follows from (ii). \square

We state a Corollary to our Main result which gives instances of uniform inertial endomorphisms, which are 0 on both A/T and T but are not finitary.

Corollary 2 *There exists an abelian group A with $r_0(A) = 1$ and p -components with order p such that the ideal of inertial endomorphisms annihilating A/T has shape $UI(A) = X + F(A)$, where X is the ideal of (all) endomorphisms annihilating both A/T and T .*

Moreover, concerning additive groups, we have $X \simeq \prod_p \mathbb{Z}(p)$ and $UI(A)/F(A)$ is an uncountable torsion-free divisible abelian group.

Proof. Consider the group A as in Proposition A of [4], which has no-critical primes. On one hand $X \subseteq UI(A)$, by that Proposition. On the other hand, by our Main Result, $UI(A)$ is the ideal of inertial endomorphisms annihilating A/T , as stated. Arguing now on T as in Remark 4 of [5], we see that any φ in $UI(A)$ acts as a finitary automorphism $\varphi|_T$ on T . Thus there is a finite set π of primes such that $\varphi|_T = m \oplus 0$ on $T = T_\pi \oplus T_{\pi'}$. Then there is C such that $A = T_\pi \oplus C$ and one can consider $\varphi_1 := m \oplus 0 \in F(A)$ where $\varphi - \varphi_1$ annihilates both T and A/T . Thus $UI(A) = X + F(A)$. Notice that $X \simeq Hom(A/T, T) \simeq \prod_p \mathbb{Z}(p)$ and $X \cap F(A) = T(X)$. \square

7 Proof of Main Result

Statement (a) follows immediately from the Characterization Theorem. Let then A have FTFR. Fix sequences $\pi_i = \{p_1, \dots, p_i\}$, (B_i) , (C_i) as in Lemma 1. Let N be the ring of mini-multiplications w.r.t. fixed sequences.

For each $\varphi \in IE(A)$, in the notation of the Characterization Theorem, $\varphi = m/n$ on $A/T(A)$ while A_π is bounded where $\pi := \pi(n)$. Moreover $A = A_\pi \oplus_\pi C_*$ where C_* is \mathbb{Q}^π -module, as $T(C_*)$ is π -divisible being π' -group and $C_*/T(C_*) \simeq A/T(A)$ is π -divisible as well. Thus there exists the semi-multiplication $0 \oplus_\pi m/n$. We may consider also $\varphi_1 := \varphi - (0 \oplus_\pi m/n)$, which is 0 on A/T . We reduced to show that if $\varphi_1 = 0$ on A/T then $\varphi_1 \in UI(A) + N$.

Apply now part (b) of the Characterization Theorem to φ_1 and fix

$$A_0 = B \oplus D \oplus C$$

and $V \leq C$ as in that statement. Then for some sufficiently large j we have $\pi(B) \cap \pi_c(A) \subseteq \pi_j \subseteq \pi_c(A)$. Fix j as well. As in Lemma 1 we have

$$A = B_j \oplus_{\pi_j} C_j.$$

Let now, for each i , $s_i := 0$ if either the p_i -component of $(D + C)/V$ is finite or that of B is finite. Otherwise, let $s_i := m_i - \alpha_i$ where $\alpha_i \in Q_{p_i}^*$ represents φ_1 on the p_i -component of $(D + C)/V$ and $m_i \in \mathbb{Z}$ represents φ_1 on the p_i -component of B . Let $s \in \mathbb{Z}$ such that the mini-multiplication $\sigma := s \oplus_{\pi_j} 0$ on $B_j \oplus_{\pi_j} C_j$ acts as multiplication by s_i on the p_i -component of B_j .

We claim that $\varphi_2 := \varphi_1 - \sigma \in UI(A)$, where it is clear that $\varphi_2 = 0$ on $V_2 := V \cap C_j$. Then let us show that φ_2 is FM on A/V_2 . Since φ_2 is inertial, it is multiplication on all but finitely many primary components of A/V_2 . Thus it is enough to show that φ_2 is FM on each primary component of A/V_2 . Fix a prime p . If $p \notin \pi_c(A)$ the statement is plain. If $p \notin \pi(B)$, the statement holds as φ_2 acts as φ_1 on the p -component of A/V_2 . Finally, if $p \in \pi_j$, consider the box decompositions:

- $A_0 = B' \oplus C'$, where B' is the p -component of B and C' is $D + C$ plus the p' -component of B ;
- $A = B'' \oplus C''$, where B'' is the p -component of B_j and C'' is C_j plus the p' -component of B_j

Note that B' and B'' (resp. C' and C'') are commensurable. This follows from the fact that $B'/(B' \cap B'')$ on one hand is a bounded p -group (as a factor of B') and on the other hand it has finite rank (as isomorphic to a subgroup of C''). For the commensurability of C' and C'' argue the same way.

Then consider $A' := (B' \cap B'') \oplus (C' \cap C'')$. Now it is enough to note that φ_2 is multiplication on the p -component of A'/V' , where $V' := V \cap A'$, to see that $\varphi_2 \in UI(A)$, as wished.

Thus we have proved that

$$IE(A) = SM(A) + UI(A) + N$$

and that $UI(A) + N$ is the ideal of inertial endomorphisms of A with periodic image. Therefore the map $\varphi \mapsto m/n$ with $\varphi = m/n$ on A/T gives a monomorphism with kernel $UI(A) + N$, while the isomorphism concerning $UI(A)/F(A)$ is stated in Proposition 3.

Finally, if A is periodic the statement is clear once one applies again Proposition 3. \square

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